Tutorial 10 : Selected problems of Assignment 10

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Announcements

(1) Extra Tutorial will be held on 3 Dec (Mon) 2:30 - 3:30 in LSB LT5.

(2) Extra office hour for Final: 4 Dec (Tue) 10:30 - 12:30

Recall the Arzelà-Ascoli theorem (for $G = (a,b) \subseteq \mathbb{R}$)

Thm $\mathcal{E} \subseteq (C[a,b], \|\cdot\|_\infty)$ is precompact (precpt) $\Leftrightarrow$ $\mathcal{E}$ is bounded (bdd) and equicontinuous (equicts)

Application to sequences: given $(f_n) \subseteq C[a,b]$ (as a seq.), $\mathcal{E} := \{f_n \mid n \in \mathbb{N}\} \subseteq C[a,b]$ (as a set)

Cor 1 If $\mathcal{E}$ is bdd and equicts, then (f_n) has a convergent subsequence (conv. subseq.)

An application of Corollary 1: Compact operator

Def Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces, a linear map $T: X \to Y$ is **compact** (cpt) if $\forall B \subseteq X$: bdd subset, $T(B) \subseteq Y$ is precpt.

Equivalently: $\forall (x_n) \subseteq X$: bdd seq, $(Tx_n) \subseteq Y$ has a conv. subseq.

Cor 2 $(X, \|\cdot\|_X) = (C[a,b], \|\cdot\|)$ w/ arbitrary norm $\|\cdot\|$, $(Y, \|\cdot\|_Y) = (C[a,b], \|\cdot\|_\infty)$

If $\forall (f_n) \subseteq (C[a,b], \|\cdot\|)$: bdd, $\{Tf_n\} \subseteq (C[a,b], \|\cdot\|_\infty)$ is bdd and equicts

then $T: C[a,b] \to C[a,b]$ is cpt.

Pf: The result follows from Cor 1.

Q1) (HW10, Q6) Define $T: C[0,1] \to C[0,1]$ by

$$T(f)(x) := \int_0^x f(t)\,dt$$

(a) Show that $T: (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ is cpt

(b) Show that $T: (C[0,1], \|\cdot\|_2) \to (C[0,1], \|\cdot\|_\infty)$ is cpt

Sol) (a) Given $(f_n) \subseteq (C[0,1], \|\cdot\|_\infty)$ bdd, $\exists M > 0$ s.t.

$$\|f_n\|_\infty \leq M, \quad \forall n \in \mathbb{N}$$

(i) $\{Tf_n\} \subseteq (C[0,1], \|\cdot\|_\infty)$ is bdd : $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, $|(Tf_n)(x)|$

$= \left|\int_0^x f_n(t)\,dt\right| \leq \|f_n\|_\infty (x-0) \leq M$, $\quad \therefore \|Tf_n\|_\infty \leq M, \ \forall n \in \mathbb{N}.$

(ii) $\{Tf_n\}$ is equicts : showing $\{Tf_n\}$ satisfies a uniform Lipschitz condition :

$\forall x, y \in [0,1]$ (WLOG $x \geq y$), $\forall n \in \mathbb{N}$, $|Tf_n(x) - Tf_n(y)| = \left|\int_y^x f_n(t)\,dt\right|$

$\leq \|f_n\|_\infty |x-y| \leq M|x-y|$

$\therefore$ By Cor 2, T is cpt.

(b) Given $(f_n) \subseteq (C[0,1], \|\cdot\|_2)$ bdd, $\exists K > 0$ s.t.

$$\|f_n\|_2^2 = \int_0^1 |f_n(t)|^2 dt \leq K, \quad \forall n \in \mathbb{N}$$

(i) $\{Tf_n\} \subseteq (C[0,1], \|\cdot\|_\infty)$ is bdd : $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$,

$$|Tf_n(x)| = \left|\int_0^x f_n(t)\, dt\right| \leq \int_0^1 |f_n(t)||1|\, dt \leq \underbrace{\left(\int_0^1 |f_n(t)|^2\right)^{\frac{1}{2}} \left(\int_0^1 |1|^2 dt\right)^{\frac{1}{2}}}_{\text{(Cauchy-Schwarz ineq.)}} \leq \sqrt{K}$$

$$\therefore \|Tf_n\|_\infty \leq \sqrt{K}, \quad \forall n \in \mathbb{N}.$$

(ii) $\{Tf_n\}$ is equicts : showing $\{Tf_n\}$ satisfies a uniform Hölder condition :

$\forall x, y \in [0,1]$ (WLOG $x \geq y$), $\forall n \in \mathbb{N}$, $|Tf_n(x) - Tf_n(y)| = \left|\int_y^x f_n(t) dt\right|$

$$\leq \int_y^x |f_n(t)||1| dt \leq \underbrace{\left(\int_y^x |f_n(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_y^x |1|^2 dt\right)^{\frac{1}{2}}}_{\text{(Cauchy-Schwarz ineq.)}} \leq \sqrt{K}\, |x-y|^{\frac{1}{2}}$$

$\therefore$ By Cor 2, T is cpt. $-\boxed{\text{Q.E.D.}}$

Q2) (HW 10, Q8) Fix $K(x,y) \in C([a,b]\times[a,b])$, $\forall f \in C[a,b]$, define the integral transform $Tf : [a,b] \to \mathbb{R}$ by $(Tf)(x) = \int_a^b K(x,y) f(y) dy$

(a) Show that T defines a linear operator $T : C[a,b] \to C[a,b]$.

(b) Show that $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ is cpt.

Sol) (a) Showing $Tf \in C[a,b]$: $\forall \varepsilon > 0$, by uniform cty of K.

$\exists \delta > 0$ s.t. $\forall x, x' \in [a,b]$ w/ $|x - x'| < \delta$, $\forall y \in [a,b]$, $|K(x,y) - K(x',y)| < \varepsilon$

$\therefore$ $|Tf(x) - Tf(x')| = \left| \int_a^b (K(x,y) - K(x',y)) f(y) dy \right| \leq \varepsilon \cdot \|f\|_\infty (b-a)$

Linearity of T follows from the linearity of integration.

(b) Given $(f_n) \subseteq (C[a,b], \|\cdot\|_\infty)$ bdd, $\exists M > 0$ s.t. $\|f_n\|_\infty \leq M$, $\forall n \in \mathbb{N}$

(i) $\{Tf_n\} \subseteq (C[a,b], \|\cdot\|_\infty)$ is bdd: $\forall n \in \mathbb{N}$, $\forall x \in [a,b]$, $|(Tf_n)(x)|$

$= \left| \int_a^b K(x,y) f_n(y) dy \right| \leq \|K\|_\infty \|f_n\|_\infty (b-a) \leq \|K\|_\infty M (b-a)$

(ii) $\{Tf_n\}$ is equicts: Under same notations as in (a), $\exists \delta > 0$, $\forall n \in \mathbb{N}$,

$$|Tf_n(x) - Tf_n(x')| \leq \varepsilon \cdot \|f_n\|_\infty (b-a) \leq \varepsilon M (b-a)$$

$\therefore$ By Cor 2, T is cpt. $\boxtimes$

Q3) (HW10, Q4) Let $\mathcal{E} = \{f_1, \cdots, f_N\} \subseteq C[a,b]$ be a finite set, show that $\mathcal{E}$ is bdd and equicts.

Sol 1) By definition: (1) $\mathcal{E}$ is bdd:

Let $M := \max_{1 \leq i \leq N} \{\|f_i\|_\infty\}$, then $\forall\, 1 \leq i \leq N$, $\|f_i\|_\infty \leq M$

(2) $\mathcal{E}$ is equicts: $\forall\, \varepsilon > 0$, since $\forall\, 1 \leq i \leq N$, f_i is uniformly cts,

$\exists\, \delta_i > 0$ s.t. $\forall\, x, y \in [a,b]$ w/ $|x-y| < \delta_i$, $|f_i(x) - f_i(y)| < \varepsilon$

Let $\delta := \min_{1 \leq i \leq N} \delta_i$, then $\forall\, x, y \in [a,b]$ w/ $|x-y| < \delta$, $\forall\, 1 \leq i \leq N$,

$$|f_i(x) - f_i(y)| < \varepsilon$$

$- \blacksquare$

Sol 2) By Thm, suffices to show $\mathcal{E}$ is precpt:

$\forall\, (g_n) \subseteq \mathcal{E}$, since $\mathcal{E}$ is finite, (g_n) contain a subsequence (g_{n_k}) of the form $g_{n_k} = f_i$, $\exists\, 1 \leq i \leq N$, $\forall\, k \in \mathbb{N}$

$\therefore (g_{n_k})$ converges uniformly to f_i, hence is convergent in $(C[a,b], \|\cdot\|_\infty)$

$- \blacksquare$